On the Existence of Unions of Timed Scenarios

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1. Background: Timed Scenarios
   - Semantics
   - Consistency
   - Distance Tables
   - Union of Timed Scenarios
2. Necessary and Sufficient Conditions for the Existence of Unions

Background: Timed Scenarios

- A formal way of specifying behaviours of a real-time system
- A timed scenario specifies all the behaviours that:
  - share a particular sequence of events;
  - satisfy the constraints on the times between events.

Scenarios: Example

$\xi = (\mathcal{E}, \mathcal{C})$

$\xi_1 = (a\ b\ c\ f, \{\tau_{0,1} \leq 5, \tau_{0,2} \leq 4\})$

is represented by

$$\xi_1 \quad \boxed{\begin{array}{ll} 0: & a; \\ 1: & b\ \{\tau_{01} \leq 5\}; \\ 2: & c\ \{\tau_{02} \leq 4\}; \\ 3: & f\ . \end{array}}$$

Semantics of Scenarios

$$\xi_1 \quad \boxed{\begin{array}{ll} 0: & a; \\ 1: & b\ \{\tau_{01} \leq 5\}; \\ 2: & c\ \{\tau_{02} \leq 4\}; \\ 3: & f\ . \end{array}}$$

$[\![\xi_1]\!]$: the set of behaviours that are allowed by ξ_1

$$[\![\xi_1]\!] = \{\ (a, t_0)(b, t_1)(c, t_2)(f, t_3) \mid \begin{array}{l} t_0 \leq t_1 \leq t_2 \leq t_3 \wedge \\ t_1 - t_0 \leq 5 \wedge \\ t_2 - t_0 \leq 4\ \} \end{array}$$

Semantics of Scenarios

$$\xi_1 \quad \boxed{\begin{array}{l} 0:\ a; \\ 1:\ b\ \{\tau_{01} \leq 5\}; \\ 2:\ c\ \{\tau_{02} \leq 4\}; \\ 3:\ f\ . \end{array}}$$

For $i < j$: $t_{ij} = t_j - t_i$

$[\![\xi_1]\!]$: the set of behaviours that are allowed by ξ_1

$$[\![\xi_1]\!] = \{\ (a, t_0)(b, t_1)(c, t_2)(f, t_3) \mid\ t_0 \leq t_1 \leq t_2 \leq t_3 \wedge t_{01} \leq 5 \wedge t_{02} \leq 4\ \}$$

Consistency of Scenarios

A scenario ξ is *consistent* iff $[\![\xi]\!] \neq \emptyset$; otherwise it is *inconsistent*.

Example:

$$\xi_1 \quad \boxed{\begin{array}{l} 0: \; a; \\ 1: \; b\;\{\tau_{01} \leq 5\}; \\ 2: \; c\;\{\tau_{02} \leq 4\}; \\ 3: \; f. \end{array}} \qquad \xi_2 \quad \boxed{\begin{array}{l} 0: \; a; \\ 1: \; b\;\{\tau_{01} \geq 2\}; \\ 2: \; c\;\{\tau_{12} \geq 2\}; \\ 3: \; f\;\{\tau_{03} \leq 2\}. \end{array}}$$

ξ_1 is consistent, while ξ_2 is inconsistent.

Upper and Lower Bounds on Time Differences

ξ: a consistent scenario of length n
For $0 \leq i < j < n$:

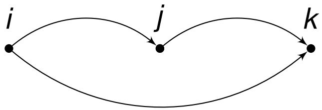
$$m_{ij}^{\xi} = min\{t_{ij}^{\mathcal{B}} \mid \mathcal{B} \in [\![\xi]\!]\}$$

$$M_{ij}^{\xi} = max\{t_{ij}^{\mathcal{B}} \mid \mathcal{B} \in [\![\xi]\!]\}$$

If there is no upper bound for some i and j, then $M_{ij}^{\xi} = \infty$.

Obviously $0 \leq m_{ij} \leq t_{ij} \leq M_{ij} \leq \infty$.

Theorem



For $0 \leq i < j < k \leq n$:

$$m_{ij} + m_{jk} \leq m_{ik} \leq \left\{ \begin{array}{c} m_{ij} + M_{jk} \\ M_{ij} + m_{jk} \end{array} \right\} \leq M_{ik} \leq M_{ij} + M_{jk} \quad (1)$$

Distance Tables

Another representation for the constraints of ξ

ξ_1

0 : a;
1 : $b\ \{\tau_{01} \leq 5\}$;
2 : $c\ \{\tau_{02} \leq 4\}$;
3 : f .

$\mathcal{D}^{\xi_1}$

	1	2	3
0	(0, 5)	(0, 4)	(0, ∞)
1		(0, ∞)	(0, ∞)
2			(0, ∞)

$l_{01} = 0$
$h_{01} = 5$

Stable Distance Tables

	1	2	3
0	(0, 5)	(0, 4)	$(0, \infty)$
1		$(0, \infty)$	$(0, \infty)$
2			$(0, \infty)$

A distance table of size n is *stable* iff

- $l_{ij} \leq h_{ij}$, for all $0 \leq i < j < n$
- for all $0 \leq i < j < k < n$,

$$l_{ij} + l_{jk} \leq l_{ik} \leq \left\{ \begin{array}{c} l_{ij} + h_{jk} \\ h_{ij} + l_{jk} \end{array} \right\} \leq h_{ik} \leq h_{ij} + h_{jk} \qquad (2)$$

Stable Distance Tables

ξ_1

$$\begin{array}{|ll|}\hline 0: & a; \\ 1: & b\ \{\tau_{01} \leq 5\}; \\ 2: & c\ \{\tau_{02} \leq 4\}; \\ 3: & f\ . \\ \hline \end{array}$$

$\mathcal{D}^{\xi_1}$	1	2	3
0	(0, 5)	(0, 4)	$(0, \infty)$
1		$(0, \infty)$	$(0, \infty)$
2			$(0, \infty)$

$\mathcal{D}^{\xi_1}$ is not stable

$h_{01} + l_{12} \leq h_{02}$

$5 + 0 \not\leq 4$

$l_{01} + h_{12} \leq h_{02}$

$0 + \infty \not\leq 4$

Stabilizing Distance Tables

$$l_{ij} + l_{jk} \leq l_{ik} \leq \left\{ \begin{array}{c} l_{ij} + h_{jk} \\ h_{ij} + l_{jk} \end{array} \right\} \leq h_{ik} \leq h_{ij} + h_{jk} \tag{2}$$

$$l_{ij} + l_{jk} > l_{ik} \longrightarrow l_{ik} := l_{ij} + l_{jk} \tag{R1}$$
$$l_{ik} > l_{ij} + h_{jk} \longrightarrow l_{ij} := l_{ik} - h_{jk} \tag{R2}$$
$$l_{ik} > h_{ij} + l_{jk} \longrightarrow l_{jk} := l_{ik} - h_{ij} \tag{R3}$$
$$l_{ij} + h_{jk} > h_{ik} \longrightarrow h_{jk} := h_{ik} - l_{ij} \tag{R4}$$
$$h_{ij} + l_{jk} > h_{ik} \longrightarrow h_{ij} := h_{ik} - l_{jk} \tag{R5}$$
$$h_{ik} > h_{ij} + h_{jk} \longrightarrow h_{ik} := h_{ij} + h_{jk} \tag{R6}$$

Apply iteratively!

- Low values increase and high values decrease.
- Termination: either (2) is satisfied or table becomes invalid i.e., $l_{ij} > h_{ij}$, for some $i < j$.

Stable Distance Tables

ξ_1

0 : a;
1 : $b\ \{\tau_{01} \leq 5\}$;
2 : $c\ \{\tau_{02} \leq 4\}$;
3 : f .

$\mathcal{D}^{\xi_1}$	1	2	3
0	(0, 5)	(0, 4)	$(0, \infty)$
1		$(0, \infty)$	$(0, \infty)$
2			$(0, \infty)$

$\mathcal{D}_s^{\xi_1}$	1	2	3
0	(0, 4)	(0, 4)	$(0, \infty)$
1		(0, 4)	$(0, \infty)$
2			$(0, \infty)$

$\mathcal{D}^{\xi_1}$ is not stable

$h_{01} + l_{12} \leq h_{02}$

$5 + 0 \nleq 4$

$h_{01} := h_{02} - l_{12} = 4 - 0$

$l_{01} + h_{12} \leq h_{02}$

$0 + \infty \nleq 4$

$h_{12} := h_{02} - l_{01} = 4 - 0$

Properties of Stable Distance Tables

- The stable distance table is unique for a given scenario.
- If $\mathcal{D}^{\xi}$ is stable, then each constraint in the table is tight:

  $l_{ij} = m^{\xi}_{ij}$ and $h_{ij} = M^{\xi}_{ij}$.
- A stable table includes all the constraints that are implied by the initial set of constraints.
- Semantically-equivalent scenarios have the same stable distance table.

Optimized Scenarios

- Minimal set of constraints
- Removal of any of the constraints would change the semantics

Explicit Constraints

If $\xi = (\mathcal{E}, \mathcal{C})$ is an optimized scenario, then the members of $\mathcal{C}$ are the explicit constraints.

η

$$
\begin{array}{l}
0 : a \,; \\
1 : b \, \{\tau_{01} \geq 6\} \,; \\
2 : c \, \{\tau_{02} \leq 8, \tau_{12} \geq 1\} \,.
\end{array}
$$

	1	2
0	(6, 7)	(7, 8)
1		(1, 2)

- $\mathcal{D}_s^\eta = \{\tau_{01} \geq 6, \tau_{01} \leq 7, \tau_{02} \geq 7, \tau_{02} \leq 8, \tau_{12} \geq 1, \tau_{12} \leq 2\}$.
- $\mathcal{C} = \{\tau_{01} \geq 6, \tau_{02} \leq 8, \tau_{12} \geq 1\}$ is the set of explicit constraints of η.

Motivation

- ξ and η:
  - two scenarios of length n with the same sequence of events, $\mathcal{E}$
  - $\forall_{0 \leq i < j < n}\ I^{\xi}_{ij} \cap I^{\eta}_{ij} \neq \emptyset$ (all intervals intersect)

Intersection

- The *intersection* of ξ and η ($\xi \cap \eta$)
  - a scenario whose sequence of events is $\mathcal{E}$ and $\mathcal{D}^{\xi \cap \eta}[i, j] = (\max(m^{\xi}_{ij}, m^{\eta}_{ij}), \min(M^{\xi}_{ij}, M^{\eta}_{ij}))$
  - $[\![\xi \cap \eta]\!] = [\![\xi]\!] \cap [\![\eta]\!]$

For example:

Motivation

$[\![\xi \cap \eta]\!] = [\![\xi]\!] \cap [\![\eta]\!]$

For union, it is not always the case that
$[\![\xi \cup \eta]\!] = [\![\xi]\!] \cup [\![\eta]\!]$

Which makes it more interesting!

Combination (Quasi-union)

- ξ and η:
  - two scenarios of length n with the same sequence of events, $\mathcal{E}$
  - $\forall_{0 \leq i < j < n}\ I_{ij}^{\xi} \cap I_{ij}^{\eta} \neq \emptyset$ (all intervals intersect)
- The *combination (quasi-union)* of ξ and η ($\xi \Cup \eta$)
  - a scenario whose sequence of events is $\mathcal{E}$ and
  - $\mathcal{D}^{\xi \Cup \eta}[i, j] = (\min(m_{ij}^{\xi}, m_{ij}^{\eta}), \max(M_{ij}^{\xi}, M_{ij}^{\eta}))$

For example:

Combination (Quasi-union): Example

ξ

0 : *a* ;
1 : *b* $\{\tau_{01} \leq 4\}$;
2 : *c* .

	1	2
0	$(0, 4)$	$(0, \infty)$
1		$(0, \infty)$

η

0 : *a* ;
1 : *b* ;
2 : *c* $\{\tau_{02} \geq 7\}$.

	1	2
0	$(0, \infty)$	$(7, \infty)$
1		$(0, \infty)$

$\xi \Cup \eta$

0 : *a* ;
1 : *b* ;
2 : *c* .

	1	2
0	$(0, \infty)$	$(0, \infty)$
1		$(0, \infty)$

Combination (Quasi-union)

$[\![\xi]\!] \cup [\![\eta]\!] \subseteq [\![\xi \Cup \eta]\!]$.
What about
$[\![\xi \Cup \eta]\!] \subseteq [\![\xi]\!] \cup [\![\eta]\!]$?

Zigzagging Behaviours

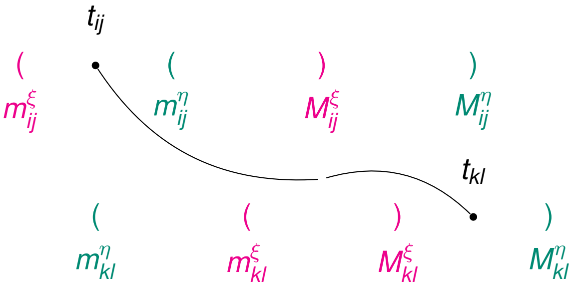
- ξ and η: two consistent scenarios, such that
- $\xi \Cup \eta$ is defined.

$[\![\xi \Cup \eta]\!] = [\![\xi]\!] \cup [\![\eta]\!] \cup \mathcal{Z}(\xi, \eta)$, where
$[\![\xi]\!] \cap \mathcal{Z}(\xi, \eta) = \emptyset$ and
$[\![\eta]\!] \cap \mathcal{Z}(\xi, \eta) = \emptyset$.

We call members of $\mathcal{Z}(\xi, \eta)$ *zigzagging behaviours*.

Zigzagging Behaviours

For example:



$t_{ij} \in I_{ij}^{\xi} \setminus I_{ij}^{\eta}$

$t_{kl} \in I_{kl}^{\eta} \setminus I_{kl}^{\xi}$

ξ and η: two scenarios of length n, $\xi \Cup \eta$ defined.

If behaviour $\mathcal{B}^z \in \mathcal{Z}(\xi, \eta)$ is such that
$t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\eta} \setminus I_{ij}^{\xi}$, $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi} \setminus I_{kl}^{\eta}$ $(i \neq k \vee j \neq l)$

Then we say
$\mathcal{B}^z$ zigzags *through* ij and kl.

Zigzagging Behaviours: Example

ξ

0 : a ;
1 : b $\{\tau_{01} \leq 4\}$;
2 : c .

	1	2
0	$(0, 4)$	$(0, \infty)$
1		$(0, \infty)$

η

0 : a ;
1 : b ;
2 : c $\{\tau_{02} \geq 7\}$.

	1	2
0	$(0, \infty)$	$(7, \infty)$
1		$(0, \infty)$

$\xi \Cup \eta$

0 : a ;
1 : b ;
2 : c .

	1	2
0	$(0, \infty)$	$(0, \infty)$
1		$(0, \infty)$

$\mathcal{B}^z = (a, 0)(b, 5)(c, 6)$

$t_{01} = 5 \in I^{\eta}_{01} \setminus I^{\xi}_{01}$

$t_{02} = 6 \in I^{\xi}_{01} \setminus I^{\eta}_{01}$

$\mathcal{B}^z$ zigzags through 01 and 02.

Union

If $\mathcal{Z}(\xi, \eta) = \emptyset$, $\xi \Cup \eta$ becomes the *union* of ξ and η.
$[\![\xi \cup \eta]\!] = [\![\xi]\!] \cup [\![\eta]\!]$

Union: Example

$\gamma = \xi \cup \eta$: $[\![\gamma]\!] = [\![\xi]\!] \cup [\![\eta]\!]$

ξ

0 : *a* ;
1 : *b* ;
2 : *c* $\{\tau_{12} \leq 6\}$;
3 : *d* $\{\tau_{03} \geq 3\}$.

	1	2	3
0	$(0, \infty)$	$(0, \infty)$	$(3, \infty)$
1		$(0, 6)$	$(0, \infty)$
2			$(0, \infty)$

η

0 : *a* ;
1 : *b* $\{\tau_{01} \leq 4\}$;
2 : *c* $\{\tau_{02} \leq 6\}$;
3 : *d* .

	1	2	3
0	$(0, 4)$	$(0, 6)$	$(0, \infty)$
1		$(0, 6)$	$(0, \infty)$
2			$(0, \infty)$

$\xi \cup \eta$

0 : *a* ;
1 : *b* ;
2 : *c* $\{\tau_{12} \leq 6\}$;
3 : *d* .

	1	2	3
0	$(0, \infty)$	$(0, \infty)$	$(0, \infty)$
1		$(0, 6)$	$(0, \infty)$
2			$(0, \infty)$

Combination (Quasi-union)

Theorem

Let $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ be two optimized scenarios of length n such that $\xi \Cup \eta$ is defined.

If $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then, there exist

a constraint $\alpha \in \mathcal{C}_1$ of the form $\tau_{ij} \sim a$
and
a constraint $\beta \in \mathcal{C}_2$ of the form $\tau_{kl} \sim b$ ($ij \neq kl$)

such that $\alpha \notin \mathcal{C}_2$ and $\beta \notin \mathcal{C}_1$.

Theorem: Example

ξ

$0 : a\,;$
$1 : b\ \{\tau_{01} \leq 4\}\ ;$
$2 : c\,.$

	1	2
0	(0, 4)	(0, ∞)
1		(0, ∞)

η

$0 : a\,;$
$1 : b\,;$
$2 : c\ \{\tau_{02} \geq 7\}\ .$

	1	2
0	(0, ∞)	(7, ∞)
1		(0, ∞)

$\mathcal{B}^z = (a, 0)(b, 5)(c, 6)$
$t_{01} = 5 \in I^{\eta}_{01} \setminus I^{\xi}_{01}$
$t_{02} = 6 \in I^{\xi}_{01} \setminus I^{\eta}_{01}$

$\mathcal{Z}(\xi, \eta) \neq \emptyset$
therefore,
$\alpha = \tau_{01} \leq 4$, $\beta = \tau_{02} \geq 7$

Combination (Quasi-union)

Recall the theorem:

Let $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ be two optimized scenarios of length n such that $\xi \Cup \eta$ is defined.

If $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then, there exist

a constraint $\alpha \in \mathcal{C}_1$ of the form $\tau_{ij} \sim a$
and
a constraint $\beta \in \mathcal{C}_2$ of the form $\tau_{kl} \sim b$ $(ij \neq kl)$

such that $\alpha \notin \mathcal{C}_2$ and $\beta \notin \mathcal{C}_1$.

The theorem provides a sufficient condition for the existence of union: if $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ do not contain such an α and β, then $\mathcal{Z}(\xi, \eta) = \emptyset$.

Combination (Quasi-union)

Recall the theorem:

Let $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ be two optimized scenarios of length n such that $\xi \Cup \eta$ is defined.

If $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then, there exist

a constraint $\alpha \in \mathcal{C}_1$ of the form $\tau_{ij} \sim a$
and
a constraint $\beta \in \mathcal{C}_2$ of the form $\tau_{kl} \sim b$ ($ij \neq kl$)

such that $\alpha \notin \mathcal{C}_2$ and $\beta \notin \mathcal{C}_1$.

But the condition is not necessary, in general:
if ξ and η contain such an α and β, then $\mathcal{Z}(\xi, \eta)$ might be empty.

Combination (Quasi-union)

But the condition is not necessary, in general:
if ξ and η contain such an α and β, then $\mathcal{Z}(\xi, \eta)$ might be empty.

ξ

0 : *a* ;
1 : *b* ;
2 : *c* $\{\tau_{02} \geq 7\}$.

	1	2
0	(0, ∞)	(7, ∞)
1		(0, ∞)

η

0 : *a* ;
1 : *b* $\{\tau_{01} \leq 8\}$;
2 : *c* $\{\tau_{02} \leq 10\}$.

	1	2
0	(0, 8)	(0, 10)
1		(0, 10)

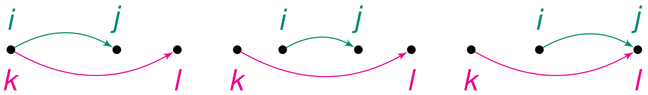
$\alpha = \tau_{02} \geq 7$, in ξ
$\beta = \tau_{01} \leq 8$, in η

However, no behaviour in $\mathcal{Z}(\xi, \eta)$.

z_pairs

ξ and η: $\xi \not\subseteq \eta$, $\eta \not\subseteq \xi$ and $\xi \Cup \eta$ defined
$\alpha = \tau_{ij} \sim a$ in ξ (not in η) $\beta = \tau_{kl} \sim b$ in η (not in ξ) $(i \neq k \vee j \neq l)$
α and β form a *z_pair* if one of the following conditions holds:

1. $0 \leq k \leq i < j \leq l < n$ and
   (a) $\alpha = \tau_{ij} \geq a$, $\beta = \tau_{kl} \geq b$, $m^{\eta}_{ij} < a$, $m^{\xi}_{kl} < b$, or



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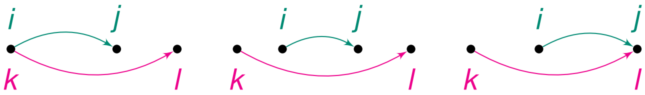
α and β form a *z_pair* if one of the following conditions holds:

1. $0 \leq k \leq i < j \leq l < n$ and
   - (a) $\alpha = \tau_{ij} \geq a$, $\beta = \tau_{kl} \geq b$, $m^{\eta}_{ij} < a$, $m^{\xi}_{kl} < b$, or
   - (b) $\alpha = \tau_{ij} \geq a$, $\beta = \tau_{kl} \leq b$, $m^{\eta}_{ij} < a$, $b < M^{\xi}_{kl}$, and additionally $M^{\xi \Cup \eta}_{ki} + a + M^{\xi \Cup \eta}_{jl} > b$, or
   - (c) $\alpha = \tau_{ij} \leq a$, $\beta = \tau_{kl} \geq b$, $a < M^{\eta}_{ij}$, $m^{\xi}_{kl} < b$, and additionally $m^{\xi \Cup \eta}_{ki} + a + m^{\xi \Cup \eta}_{jl} < b$, or
   - (d) $\alpha = \tau_{ij} \leq a$, $\beta = \tau_{kl} \leq b$, $a < M^{\eta}_{ij}$, $b < M^{\xi}_{kl}$.



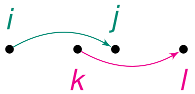
z_pairs

ξ and η: $\xi \not\subseteq \eta$, $\eta \not\subseteq \xi$ and $\xi \Cup \eta$ defined

$\alpha = \tau_{ij} \sim a$ in ξ (not in η) $\beta = \tau_{kl} \sim b$ in η (not in ξ) $(i \neq k \vee j \neq l)$

α and β form a *z_pair* if one of the following conditions holds:

1. $0 \leq i < k < j < l < n$ and
   - (a) $\alpha = \tau_{ij} \geq a$, $\beta = \tau_{kl} \geq b$, $m^{\eta}_{ij} < a$, $m^{\xi}_{kl} < b$, and additionally $m^{\xi \Cup \eta}_{il} - a < b - m^{\xi \Cup \eta}_{kj}$, or
   - (b) $\alpha = \tau_{ij} \geq a$, $\beta = \tau_{kl} \leq b$, $m^{\eta}_{ij} < a$, $b < M^{\xi}_{kl}$, and additionally $a + M^{\xi \Cup \eta}_{jl} > m^{\xi \Cup \eta}_{ik} + b$, or
   - (c) $\alpha = \tau_{ij} \leq a$, $\beta = \tau_{kl} \geq b$, $a < M^{\eta}_{ij}$, $m^{\xi}_{kl} < b$, and additionally $a + m^{\xi \Cup \eta}_{jl} < M^{\xi \Cup \eta}_{ik} + b$, or
   - (d) $\alpha = \tau_{ij} \leq a$, $\beta = \tau_{kl} \leq b$, $a < M^{\eta}_{ij}$, $b < M^{\xi}_{kl}$, and additionally $M^{\xi \Cup \eta}_{il} - a > b - M^{\xi \Cup \eta}_{kj}$.



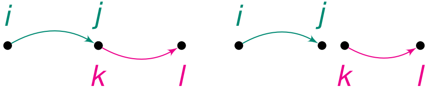
z_pairs

ξ and η: $\xi \not\subseteq \eta$, $\eta \not\subseteq \xi$ and $\xi \Cup \eta$ defined

$\alpha = \tau_{ij} \sim a$ in ξ (not in η) $\beta = \tau_{kl} \sim b$ in η (not in ξ) $(i \neq k \vee j \neq l)$

α and β form a *z_pair* if one of the following conditions holds:

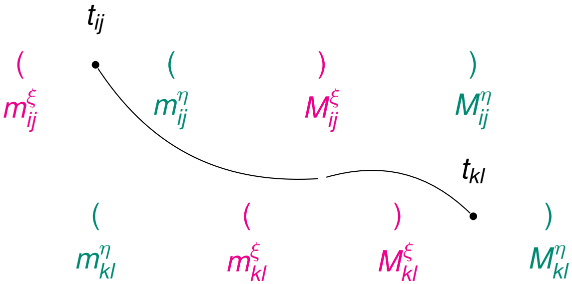
1. $0 \leq i < j \leq k < l < n$ and
   - (a) $\alpha = \tau_{ij} \geq a$, $\beta = \tau_{kl} \geq b$, $m^{\eta}_{ij} < a$, $m^{\xi}_{kl} < b$, or
   - (b) $\alpha = \tau_{ij} \geq a$, $\beta = \tau_{kl} \leq b$, $m^{\eta}_{ij} < a$, $b < M^{\xi}_{kl}$, or
   - (c) $\alpha = \tau_{ij} \leq a$, $\beta = \tau_{kl} \geq b$, $a < M^{\eta}_{ij}$, $m^{\xi}_{kl} < b$, or
   - (d) $\alpha = \tau_{ij} \leq a$, $\beta = \tau_{kl} \leq b$, $a < M^{\eta}_{ij}$, $b < M^{\xi}_{kl}$.



z_pairs

The conditions capture all the possibilities for $I^{\eta}_{ij} \setminus I^{\xi}_{ij} \neq \emptyset$ and $I^{\xi}_{kl} \setminus I^{\eta}_{kl} \neq \emptyset$, to guarantee "there is room" for behaviours to zigzag through *ij* and *kl*.

The *additional* conditions specify certain relations that must hold between various minima and maxima in ξ and η for there to be zigzagging behaviours.



z_pair Might Not Be Between Explicit Constraints

$0 : a$
$1 : b\ \{\tau_{01} \geq 1\}$
$2 : c\ \{\tau_{02} \leq 2\}$
$3 : d$

ξ

	1	2	3
0	$(1,2)$	$(1,2)$	$(1,\infty)$
1		$(0,1)$	$(0,\infty)$
2			$(0,\infty)$

$0 : a$
$1 : b\ \{\tau_{01} \leq 1\}$
$2 : c\ \{\tau_{12} \leq 1\}$
$3 : d\ \{\tau_{13} \geq 4\}$

η

	1	2	3
0	$(0,1)$	$(0,2)$	$(4,\infty)$
1		$(0,1)$	$(4,\infty)$
2			$(3,\infty)$

- $\tau_{02} \leq 2$ of ξ and $\tau_{13} \geq 4$ of η are different *explicit* constraints.
  - But they do not form a *z_pair*:
    case (2)(c) of definition of *z_pair*:
    However $a = 2 \nless M^{\eta}_{02} = 2$.

z_pair Between Implied Constraints

$0 : a$
$1 : b\ \{\tau_{01} \geq 1\}$
$2 : c\ \{\tau_{02} \leq 2\}$
$3 : d$

ξ

	1	2	3
0	$(1, 2)$	$(1, 2)$	$(1, \infty)$
1		$(0, 1)$	$(0, \infty)$
2			$(0, \infty)$

$0 : a$
$1 : b\ \{\tau_{01} \leq 1\}$
$2 : c\ \{\tau_{12} \leq 1\}$
$3 : d\ \{\tau_{13} \geq 4\}$

η

	1	2	3
0	$(0, 1)$	$(0, 2)$	$(4, \infty)$
1		$(0, 1)$	$(4, \infty)$
2			$(3, \infty)$

$\tau_{02} \geq 1$ of ξ (an *implied* constraint) and $\tau_{13} \geq 4$ of η form a *z_pair*.

A Necessary Condition for the Non-existence of Union

no union $\equiv$ zigzagging $\Rightarrow$ *z_pair*

Theorem

ξ and η: two scenarios of length n, $\xi \not\subseteq \eta$ and $\eta \not\subseteq \xi$,
$\xi \Cup \eta$ is defined

If $\mathcal{B}^z \in \mathcal{Z}(\xi, \eta)$ zigzags through some ij and kl, then
there exist $\alpha = \tau_{ij} \sim a$ in ξ and $\beta = \tau_{kl} \sim b$ in η
such that
α and β form a z_pair.

Consequence of the Theorem

no union $\equiv$ zigzagging $\Rightarrow$ *z_pair*

if there is no *z_pair* between scenarios ξ and η, then $\mathcal{Z}(\xi, \eta) = \emptyset$, therefore $\xi \cup \eta$ exists.

A Sufficient Condition for the Non-existence of Union

z_pair $\Rightarrow$ zigzagging $\equiv$ no union

Theorem

ξ and η: two scenarios of length n, $\xi \not\subseteq \eta$, $\eta \not\subseteq \xi$
$\xi \Cup \eta$ is defined

If there are
$\alpha = \tau_{ij} \sim a$ in ξ and $\beta = \tau_{kl} \sim b$ in η such that α and β form a z_pair, then
there is a $\mathcal{B}^z \in \mathcal{Z}(\xi, \eta)$, such that $\mathcal{B}^z$ zigzags through ij and kl.

The Consequence of the Theorem

z_pair $\Rightarrow$ zigzagging $\equiv$ no union

if ξ and η have constraints that form a *z_pair*, then $\xi \cup \eta$ does not exist.

Example

$0 : a\,;$
$1 : b\,\{\tau_{01} \geq 3\}\,;$
$2 : c\,\{\tau_{02} \leq 7\}\,.$

ξ

	1	2
0	$(3,7)$	$(3,7)$
1		$(0,4)$

$0 : a\,;$
$1 : b\,;$
$2 : c\,\{\tau_{12} \geq 4\}\,.$

η

	1	2
0	$(0,\infty)$	$(4,\infty)$
1		$(4,\infty)$

$\xi \Cup \eta$

$0 : a\,;$
$1 : b\,;$
$2 : c\,\{\tau_{02} \geq 3\}\,.$

	1	2
0	$(0,\infty)$	$(3,\infty)$
1		$(0,\infty)$

$\mathcal{B}^z = (a,0)(b,1)(c,4) \in \mathcal{Z}(\xi,\eta)$:
$t_{01}^{\mathcal{B}^z} = 1 - 0 = 1 \notin I_{01}^{\xi}$ and $t_{12}^{\mathcal{B}^z} = 4 - 1 = 3 \notin I_{12}^{\eta}$.
According to the theorem *z_pair* exist: $\alpha = \tau_{01} \geq 3$ in ξ and $\beta = \tau_{12} \geq 4$ in η.

Union does not exist.

Summary of the Results

The existence of *z_pair*s is both a necessary and sufficient condition for the non-existence of union:

z_pair $\equiv$ zigzagging $\equiv$ no union

Therefore:

no *z_pair* $\equiv$ union exists

Conclusions

- We investigate the conditions under which $[\![\xi]\!] \cup [\![\eta]\!]$ can be represented by a single scenario, namely the union $\xi \cup \eta$.
- Our investigation reveals that in the presence of zigzagging behaviours the constraints of ξ and η must satisfy certain additional criteria.
- Based on this observation we formulate a sufficient and necessary condition for the existence of the union.

Thank You!